Reading project report: Point-free topology[1]

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Abstract

We study locales in the context of point-free topology. After the basic notions of locales and localic maps are introduced, we show that there is an adjunction between the category of locales and the category of topological spaces. Furthermore, we show that this adjunction restricts to the equivalence of categories of spatial locales and sober spaces. In the last section, we briefly discuss the structure of sublocales of a locale.

1 Introduction

A great deal of information about a topological space is carried by its lattice of open sets. For instance, the notions of compactness, continuous maps and that of sheaves on a topological space refer only to open subsets without ever mentioning points of a space. The question then arises: exactly how much information does the lattice of opens contain about a space? This is the question we are aiming to answer in this project report.

Our approach to point-free topology is via the study of *locales*. A locale is a complete lattice whose finite meets distribute over arbitrary joins, in the same vein that finite intersections of open sets distribute over arbitrary unions of open sets. Thus a locale is an algebraic 'model' for a lattice of opens of a topological space. However, while for every topological space its lattice of

[1]This report is an outcome of an individual reading project I did as a part of MSc Logic degree at the ILLC, University of Amsterdam. The project was supervised by Nick Bezhanishvili.

opens is a locale, not every locale corresponds to a lattice of opens. Localic topology may thus be seen as a generalisation of classical topology, which allows one to study more exotic space-like structures. One must nonetheless keep in mind that in localic topology, in contrast to its classical counterpart, we are only allowed to talk about properties expressible using the generalised open sets, that is, elements of a locale. While we will reconstruct the notion of a point (Definition 20), this is a derived notion, and there is no guarantee that a locale has points to begin with.

A philosophical motivation for studying pointless locales comes from the fact that they are closer to our intuitive or practical notion of space. The idea here is that any notion of space encountered in 'real life' does not (at least *a priori*) consist of points, but rather of 'places', parts of space that somehow glue together to form the entirety of the space. It is these 'realistic places of non-trivial extent' ([5, Preface]) that localic topology takes as primitives, and this viewpoint will be our conceptual guideline throughout the report. A strong motivating example of this phenomenon may be found in quantum mechanics, where it might not make sense to assign a definite location to a particle, while it still occupies a certain region in space, albeit with no sharp boundaries. This idea has been developed in Heunen, Landsman and Spitters [1].

A more mathematical motivation comes from the, somewhat surprising, fact that some central results that rely on choice in classical topology are not just true in localic topology, but are also constructive. Examples of such results are the Stone-Čech compactification for locales and the fact that a product of compact locales is compact ([3, p. 48], [5, Preface]), at least when the locales are coherent (Tychonoff's theorem in classical topology). Here, however, we will not get this far. The reader is referred to Johnstone ([3] and [2]) for an overview of the subject from this point of view.

The report is structured as follows. In Section 2 we cover some results and definitions that will be used throughout the paper. We assume some familiarity with lattices and category theory, especially the theory of adjoint functors. The single most important result in this section is the Adjoint functor theorem for posets 7, which we prove in detail, as more or less everything else builds on that. Section 3 discusses the category **Loc** and other basic notions concerning locales, including construction of locales from topological spaces. In Section 4 we discuss different incarnations of the notion of a point of a locale, as well as answer the question posed in the beginning about the lattice of opens. We do the latter by constructing an adjunction between the categories of locales and topological spaces. We further show that this adjunction restricts to an equivalence of categories of *spatial locales* (Definition 18) and *sober spaces* (Definition 4). Thus as long as the space is sober, its lattice of opens in fact carries *all* information about the space, in the sense that we may reconstruct the space (up to an isomorphism) from its lattice of opens. Section 5 defines sublocales (vaguely corresponding to subspaces) and shows how the notion of a closed and open subspace carries over to sublocales.

The main source for this report are the first three chapters of *Frames and Locales* by Picado and Pultr [5].

Remark 1. We make a convention for the entire report that *every topological space is T_0 (Kolmogorov)*. Thus, for example, when we speak of the category of topological spaces **Top**, what we really mean is the category of all T_0 spaces.

Remark 2. We think of posets and lattices as categories in the standard way, with joins being the coproduct and meets the products. This is a useful perspective especially when defining the localic maps and working with the Heyting operation $\to$, which we think of as the right adjoint to the product functor:

$$a \wedge - \;\dashv\; a \to -.$$

We will exploit this perspective without further notice.

2 Preliminary notions

We introduce some aspects of category and lattice theories needed for the presentation.

2.1 Lattices and sober spaces

Definition 3 (Meet-irreducible element)**.** An element $m \in L$ in a lattice is called *meet-irreducible* if $m \neq 1$ (if the top element exists) and whenever $a \wedge b \leq m$, then either $a \leq m$ or $b \leq m$.

An example of a class of meet-irreducible elements with a nice description is given by topological spaces. Let X be a topological space and denote its lattice of open sets by $\Omega(X)$. Then for each $x \in X$, the open set $X \setminus \overline{\{x\}}$

is a meet-irreducible element of $\Omega(X)$. Indeed, if $U, V \in \Omega(X)$ are such that $U \cap V \subseteq X \setminus \overline{\{x\}}$, then $x \in \overline{\{x\}} \subseteq U^c \cup V^c$, so that either $x \in U^c$ or $x \in V^c$. Suppose without loss of generality that $x \in U^c$, so that $\overline{\{x\}} \subseteq U^c$, or equivalently, $U \subseteq X \setminus \overline{\{x\}}$, as required.

If each meet-irreducible element is of this form, the space X may be recovered up to an isomorphism from its lattice of open sets (as we will do in Section 4). This motivates making this into a definition.

Definition 4 (Sober space)**.** A topological space X is said to be *sober* if all meet-irreducible elements in the lattice of open sets $\Omega(X)$ are of the form $X \setminus \overline{\{x\}}$ for some $x \in X$.

Sober spaces may be equivalently characterised by completely prime filters; we defer the statement and proof of this characterisation to Section 4 (Corollary 25).

Remark 5. Sobriety is not a separation axiom. While any T_2 (Hausdorff) space is sober, sobriety is independent from T_1. Indeed, suppose a space X is Hausdorff, and let $U \in \Omega(X)$ be meet-irreducible. Since $U \neq X$ we have $U^c \neq \varnothing$, so that we may suppose towards a contradiction that we have $x, y \in U^c$ with $x \neq y$. Using T_2, let V_x and V_y be disjoint opens containing x and y, respectively. Then $V_x \cap V_y = \varnothing \subseteq U$, whence either $V_x \subseteq U$ or $V_y \subseteq U$, which is a contradiction. Thus $x = y$ for any elements in the complement, so that $U^c = \{x\}$ for some $x \in X$, whence $U = X \setminus \{x\}$.

Now consider the space $S \coloneqq \{x, y\}$, where the opens are $\varnothing, S$ and $\{x\}$ (the Sierpiński space). It is clearly not T_1, as $\{x\}$ is not closed. However, S is sober, as $\varnothing = S \setminus \overline{\{x\}}$ and $\{x\} = S \setminus \{y\}$ (and these are the only meet-irreducibles in $\Omega(S)$). Thus sobriety does not imply T_1.

Conversely, consider the space $\mathbb{N}$, where a set $U \subseteq \mathbb{N}$ is open iff $U = \varnothing$ or U^c is finite. Then each singleton is closed, so that the space is T_1. However, it is not sober. Indeed, $\varnothing$ is meet-irreducible: if $U \cap V = \varnothing$ such that both U and V are non-empty, then $U^c \cup V^c = \mathbb{N}$, which is a contradiction, as both U^c and V^c are finite, whence we must have that either $U = \varnothing$ or $V = \varnothing$. Certainly $\varnothing$ is not of the form $\mathbb{N} \setminus \{n\}$ for any $n \in \mathbb{N}$. Thus T_1 does not imply sobriety.

2.2 Adjoints and posets

The following is a general result in category theory, the proof may be found e.g. in Appendix 1 of Leinster [4].

Theorem 6 (General adjoint functor theorem)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F has a left adjoint, then it preserves limits. Moreover, in case $\mathcal{C}$ is complete and locally small, and for each $D \in \mathcal{D}$ the comma category $(D \Rightarrow F)$ has a weakly initial set, F preserves limits* if and only if *it has a left adjoint.*

The following result follows by specialising Theorem 6 to posets and noticing that the locally smallness and weak initiality conditions in the 'moreover' -part trivialise for small categories. However, we give a direct, order-theoretic proof of this.

Theorem 7 (Adjoint functor theorem for posets)**.** *Let P and Q be posets, and $f : P \to Q$ a functor (order-preserving map). If f has a left adjoint, then it preserves limits (infima). Moreover, in case P is complete, f preserves infima* if and only if *it has a left adjoint.*

Proof. First suppose f has a left adjoint $g : Q \to P$, so that for all $x \in Q$ and $y \in P$ we have

$$g(x) \leq y \text{ iff } x \leq f(y). \tag{8}$$

Let $A \subseteq P$ be a subset such that $\bigwedge A$ exists. We claim that $f\left(\bigwedge A\right)$ is the infimum of

$$fA := \{f(a) : a \in A\}.$$

Indeed, observe that

$$\begin{aligned} x \leq f\left(\bigwedge A\right) &\text{ iff } g(x) \leq \bigwedge A \\ &\text{ iff } g(x) \leq a \forall a \in A \\ &\text{ iff } x \leq f(a) \forall a \in A, \end{aligned}$$

whence $f\left(\bigwedge A\right) = \bigwedge (fA)$.

Now suppose P is complete and f preserves infima. We define a map $g : Q \to P$ by

$$x \mapsto \bigwedge \{z \in P : x \leq f(z)\}.$$

Observe that this assignment is indeed functorial (i.e. g is order-preserving). We claim that g is the sought-after left adjoint, that is, equation (8) holds. First suppose $g(x) \leq y$ for some $x \in Q$ and $y \in P$. Since f is order-preserving, we get $fg(x) \leq f(y)$, and since f preserves infima we have

$$fg(x) = \bigwedge \{f(z) : z \in P \text{ and } x \leq f(z)\} \geq x,$$

as x is a lower bound of the set we take the infimum of. Hence we have shown $x \leq f(y)$. Conversely, suppose $x \leq f(y)$, so that $g(x) \leq gf(y)$. But

$$gf(y) = \bigwedge\{z \in P : f(y) \leq f(z)\} \leq y,$$

as the infimum is in particular a lower bound. Thus $g(x) \leq y$, concluding the proof. $\square$

The following is the dual statement of Theorem 7, to which we will also refer as the Adjoint functor theorem for posets. Note that we exploit the fact that a poset is complete if and only if it is cocomplete.

Corollary 9. *Let Q and P be posets, and $g : Q \to P$ a functor (order-preserving map). If g has a right adjoint, then it preserves colimits (suprema). Moreover, in case Q is complete, g preserves suprema* if and only if *it has a right adjoint.*

The following is a straightforward exercise in basic category theory.

Proposition 10. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and*

$$\Lambda : \mathcal{C} \rightleftarrows \mathcal{D} : \Gamma$$

an adjunction (left adjoint on the left) with unit η and counit ϵ. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ containing those objects $C \in \mathcal{C}$ for which η_C is an isomorphism, and dually, let $\mathcal{D}_0$ contain those $D \in \mathcal{D}$ for which ϵ_D is an isomorphism. Then the adjunction $(\Lambda, \Gamma, \eta, \epsilon)$ restricts to an equivalence of $\mathcal{C}_0$ and $\mathcal{D}_0$.

Albeit the proof of this being straightforward, this result has profound consequences in particular cases, of which we shall see one in Section 4.

3 Locales

We define *locales*, the central objects of interest in the lattice-based approach to topology. We proceed to complete these to a category **Loc** by defining localic maps. We provide some motivation for the definitions and see what they correspond to in standard topology.

Definition 11 (Locale)**.** A *locale* L is a complete lattice whose finite meets distribute over arbitrary joins. More precisely, we have

$$\left(\bigvee A\right) \wedge b = \bigvee \{a \wedge b : a \in A\} \tag{12}$$

for all $b \in L$ and all subsets A of L.

Note that the distributivity condition (12) models the situation in the lattice of open sets $\Omega(X)$ of a topological space X, where finite meets (intersections) distribute over joins (unions). Thus for any topological space X, the poset $\Omega(X)$ is a locale whose lattice operations are given by unions and interior of intersections.

Proposition 13. *A lattice L is a locale if and only if it is a complete Heyting algebra.*

Proof. Observe that equation (12) says exactly that each product functor $b \wedge -$ preserves joins. Under the assumption of completeness, by the Adjoint functor theorem for posets (specifically Corollary 9), this occurs if and only if the product functor has a right adjoint, precisely what is needed for a lattice to be a Heyting algebra. □

Remark 14. From (the dual of) the proof of the Adjoint functor theorem for posets 7 we obtain that the Heyting operation induced by a locale is given by

$$b \to c = \bigvee \{a : a \wedge b \leq c\}.$$

Definition 15 (Localic map)**.** An order-preserving map (functor) $f : L \to M$ between locales L and M is *localic* if it preserves all meets and its left adjoint $f^* : M \to L$ preserves finite meets.

We denote the category of locales and localic maps by **Loc**.

Remark 16. Note that a localic map is not required to preserve the Heyting implication, and indeed, in general, it will not.

The following is a useful characterisation of isomorphisms in **Loc**.

Proposition 17. *Let $f : L \to K$ be a localic map, denote its left adjoint by f^*. Then the following are equivalent:*

(1) f is an isomorphism,

(2) f is a bijection of sets[1]*,*

(3) $f^* = f^{-1}$,

(4) $f \dashv f^*$.

Proof. ((1)) $\implies$ ((2)) is immediate.

((2)) $\implies$ ((3)): Let f^{-1} be the inverse of f (in **Set**). Since for all $y \in K$ we have $f^*(y) \leq f^*(y)$, we get $y \leq ff^*(y)$. On the other hand, $y = ff^{-1}(y)$, whence $f^*(y) \leq f^{-1}(y)$. Since f is order-preserving, we obtain that $ff^*(y) \leq y$, so that in fact $ff^*(y) = y$. Thus we also get for $a \in L$ that

$$a = f^{-1}f(a) = f^{-1}ff^*f(a) = f^*f(a),$$

whence by uniqueness of inverses $f^* = f^{-1}$.

((3)) $\implies$ ((4)): We immediately have $f(a) \leq y$ iff $a \leq f^*(y)$ as f^* is an order-preserving inverse of f.

((4)) $\implies$ ((1)): If $f \dashv f^*$, then f^* is localic, as its left adjoint preserves (finite) meets. Since we also have $f^* \dashv f$, from the two adjoint equations we obtain that $ff^* = \mathrm{id}_Y$ and $f^*f = \mathrm{id}_X$. □

Locales isomorphic to the lattice of opens of some topological space have a special status in the category of locales, as will be discussed in Section 4. For now, we make a definition.

Definition 18 (Spatial locale)**.** A locale is called *spatial* if it is isomorphic to the locale $\Omega(X)$ for some topological space X.

For a more internal characterisation of spatial locales, see Corollary 34.

Note that any continuous map between topological spaces $f : X \to Y$ induces the left adjoint of a localic map $f^{-1} : \Omega(Y) \to \Omega(X)$ (the preimage), as preimages preserve intersections. It is a left adjoint since preimages preserve unions (joins), so that by the Adjoint functor theorem for posets (Corollary 9) it has a right adjoint $f_* : \Omega(X) \to \Omega(Y)$ given by

$$U \mapsto \bigcup \{V \in \Omega(Y) : f^{-1}(V) \subseteq U\},$$

[1]Precisely, $Uf : UL \to UK$ is an isomorphism in **Set**, where $U : \mathbf{Loc} \to \mathbf{Set}$ is the forgetful functor.

which by the same theorem is guaranteed to preserve all meets and is thus a localic map. Note that the condition $f^{-1}(V) \subseteq U$ may be rewritten as $f(U^c) \subseteq V^c$. Thus using involutiveness of complements we observe that the effect of f_* may be written as

$$f_*(U) = \left(\overline{f(U^c)}\right)^c.$$

This is in fact the motivating example for the definition of a localic map. Furthermore, note that we have defined a functor:

$$\begin{aligned} \Omega : \mathbf{Top} &\to \mathbf{Loc} \\ X &\mapsto \Omega(X) \\ \left(X \xrightarrow{f} Y\right) &\mapsto \left(\Omega(X) \xrightarrow{f_*} \Omega(Y)\right). \end{aligned}$$

It will be the subject of Section 4 to define a right adjoint of this functor and explore the properties of the adjunction.

Remark 19. The category dual to **Loc** is called the *category of frames* and its morphisms are referred to as *frame homomorphisms*. While this dual perspective proves useful in certain contexts, in this report we will work exclusively with locales for the sake of conciseness.

4 Points and the spectrum adjunction

In Section 3 we defined a functor $\Omega : \mathbf{Top} \to \mathbf{Loc}$ turning topological spaces into locales. The natural question to ask is whether a locale gives a rise to a space. The answer is indeed yes, and our first aim is to construct a functor $\Sigma : \mathbf{Loc} \to \mathbf{Top}$. To this end, we will need the notion of a *point* capturing the intuitive idea of smaller and smaller 'approximations' of an infinitely tiny region of space (point), or a sequence of 'measurements' which are getting more and more precise. We then proceed to show that Σ is in fact the right adjoint of Ω, and that the adjunction restricts to an equivalence of categories between sober spaces and spatial locales.

First, we observe that the category **Loc** has a terminal object $\mathbf{2} \coloneqq \{0 < 1\}$ (the two-element Boolean algebra). For suppose L is a locale; we define $f : L \to \mathbf{2}$ by

$$x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $f^* : \mathbf{2} \to L$ by $f^*(1) = 1$ and $f^*(0) = 0$. It is now straightforward that f^* preserves (finite) meets and we have $f^* \dashv f$. Thus f is a localic map. For uniqueness, suppose we have another such adjunction $g^* \dashv g$ where g^* preserves finite meets. But then $g^*(1) = 1$ and $g^*(0) = 0$ since left adjoints preserve joins, whence $g^* = f^*$, and by uniqueness of adjoints $g = f$.

Therefore, following the usual notion of an 'element' of an object in a category, we define:

Definition 20 (Point)**.** A *point* of a locale L is a localic map $\mathbf{2} \to L$.

In fact we have a functor

$$\begin{aligned} P : \mathbf{Loc} &\to \mathbf{Set} \\ L &\mapsto P(L) \coloneqq \mathbf{Loc}(\mathbf{2}, L) \\ \left(L \xrightarrow{f} K\right) &\mapsto \left(P(L) \xrightarrow{f\circ -} P(K)\right). \end{aligned}$$

sending each locale to the set of its points (this is just the covariant representable functor). It turns out that for any locale L, there is a natural one-to-one correspondence between points, completely prime filters and meet-irreducible elements of L. We next make this precise.

Definition 21 (Completely prime filter)**.** A proper filter $F \subseteq L$ in a complete lattice L is *completely prime* if whenever we have $A \subseteq L$ such that $\bigvee A \in F$, then there is an $a \in A$ such that $a \in F$.

The following two propositions are now immediate from the fact that the left adjoint f^* of a localic map preserves joins and finite meets.

Proposition 22. *Let $f : L \to K$ be a localic map and $F \subseteq L$ a completely prime filter in L. Then*

$$(f^*)^{-1}F = \{k \in K : f^*(k) \in F\}$$

is a completely prime filter in K.

Proposition 23. *Localic maps send meet-irreducible elements to meet-irreducible elements.*

Given a locale L, let us denote the sets of its completely prime filters and meet-irreducible elements by $C(L)$ and $M(L)$, respectively. Using the above propositions, we may extend C and M to functors as follows.

$$\begin{aligned} C : \mathbf{Loc} &\to \mathbf{Set} \\ L &\mapsto C(L) \\ \left(L \xrightarrow{f} K\right) &\mapsto \left(C(L) \xrightarrow{(f^*)^{-1}} C(K)\right), \end{aligned}$$

where $(f^*)^{-1}$ sends each completely prime filter F to its preimage $(f^*)^{-1}F$.

$$\begin{aligned} M : \mathbf{Loc} &\to \mathbf{Set} \\ L &\mapsto M(L) \\ \left(L \xrightarrow{f} K\right) &\mapsto \left(M(L) \xrightarrow{f|_{M(L)}} M(K)\right). \end{aligned}$$

We now have the following characterisation of points.

Theorem 24. *Let L be a locale. There is a natural bijection between any two of the following:*

- *points of L,*
- *completely prime filters in L,*
- *meet-irreducible elements of L.*

More precisely, the functors P, C and M are naturally isomorphic.

Proof. We construct natural isomorphisms $\eta : P \to C$ and $\mu : C \to M$.

Define $\eta_L : \mathbf{Loc}(\mathbf{2}, L) \to C(L)$ by

$$p \mapsto (p^*)^{-1}(1),$$

which is well-defined by Proposition 22. To see that this defines a natural transformation, given a localic map $f : L \to K$, we need to show that the diagram

$$\begin{tikzcd} \mathbf{Loc}(\mathbf{2}, L) \arrow[r, "f\circ-"] \arrow[d, "\eta_L"'] & \mathbf{Loc}(\mathbf{2}, K) \arrow[d, "\eta_K"] \\ C(L) \arrow[r, "(f^*)^{-1}"'] & C(K) \end{tikzcd}$$

commutes. But this amounts to showing

$$((fp)^*)^{-1}(1) = (f^*)^{-1}((p^*)^{-1}(1))$$

for each $p \in \mathbf{Loc}(\mathbf{2}, L)$, which is immediate from $(fp)^* = p^* f^*$ and $(p^* f^*)^{-1} = (f^*)^{-1}(p^*)^{-1}$.

To see that each η_L is a bijection, define $\tilde{\eta}_L : C(L) \to \mathbf{Loc}(\mathbf{2}, L)$ by

$$F \mapsto p,$$

where $p : \mathbf{2} \to L$ is the right adjoint to $p^* : L \to \mathbf{2}$ defined by $p^*(x) = 1$ iff $x \in F$. We now have

$$\begin{aligned} \tilde{\eta}_L \eta_L(p) &= \tilde{\eta}_L((p^*)^{-1}) = p, \\ \eta_L \tilde{\eta}_L(F) &= F. \end{aligned}$$

Since $\tilde{\eta}_L$ is a pointwise inverse to a natural transformation, it is itself natural, whence we conclude that η_L is a natural isomorphism.

Next, define $\mu_L : C(L) \to M(L)$ by

$$F \mapsto \bigvee \{x \in L : x \notin F\},$$

which is well-defined, as each $\mu_L(F)$ is meet irreducible. To see this, first observe that $\mu_L(F) \notin F$ since F is completely prime. Then, indeed, if $a \wedge b \leq \mu_L(F)$, we have either $a \notin F$ or $b \notin F$ (if both are in F, then so is the meet hence also $\mu_L(F)$), without loss of generality say $a \notin F$, but then $a \leq \mu_L(F)$, as required. For naturality, given a localic map $f : L \to K$, we need to show that

$$\begin{tikzcd} C(L) \arrow[r, "(f^*)^{-1}"] \arrow[d, "\mu_L"'] & C(K) \arrow[d, "\mu_K"] \\ M(L) \arrow[r, "f|_{M(L)}"'] & M(K) \end{tikzcd}$$

commutes. This amounts to showing that for each $F \in C(L)$ we have

$$f(\mu_L(F)) = f\left(\bigvee \{x \in L : x \notin F\}\right) = \bigvee \{y \in K : y \notin (f^*)^{-1}F\}.$$

Thus suppose $y \notin (f^*)^{-1}F$, that is, $f^*(y) \notin F$. But then

$$f^*(y) \leq \bigvee \{x \in L : x \notin F\},$$

which occurs if and only if $y \leq f(\mu_L(F))$. Thus $f(\mu_L(F))$ is an upper bound for the set $\{y \in K : y \notin (f^*)^{-1}F\}$. To see that it is the least upper bound, let $u \in K$ be an upper bound, in other words for all $y \in K$ we have

$$f^*(y) \notin F \implies y \leq u.$$

Using the fact that f^* is a left adjoint to f, we get that

$$f^* f(\mu_L(F)) \leq \mu_L(F),$$

so that $f^* f(\mu_L(F)) \notin F$ (as otherwise $\mu_L(F) \in F$). But then, by assumption, $f(\mu_L(F)) \leq u$, so that $f(\mu_L(F))$ is indeed the least upper bound, and we have the desired equality.

To see that each μ_L is a bijection, define $\tilde{\mu}_L : M(L) \to C(L)$ by

$$m \mapsto \{x \in L : x \nleq m\}.$$

Observe that $\tilde{\mu}_L(m)$ is indeed a filter: it is proper as $0 \leq m$, it is a filter since m is meet-irreducible, and it is completely prime by properties of joins. We now have:

$$\tilde{\mu}_L \mu_L(F) = \left\{ x \in L : x \nleq \bigvee \{y \in L : y \notin F\} \right\}.$$

Observe that $x \notin F$ iff $x \leq \bigvee \{y \in L : y \notin F\}$ iff $x \notin \tilde{\mu}_L \mu_L(F)$, whence $\tilde{\mu}_L \mu_L(F) = F$. On the other hand,

$$\begin{aligned} \mu_L \tilde{\mu}_L(m) &= \bigvee \{x \in L : x \notin \{y \in L : y \nleq m\}\} \\ &= \bigvee \{x \in L : x \leq m\} \\ &= m, \end{aligned}$$

so that $\tilde{\mu}_L$ is indeed a pointwise inverse to μ_L, so that we indeed have a natural isomorphism, as before. □

We may now characterise sober spaces (Definition 4) in terms of completely prime filters. To this end, observe that for any topological space X and any $x \in X$, the set $\{U \in \Omega(X) : x \in U\}$ is a completely prime filter in $\Omega(X)$ (it is clearly a filter and complete primeness follows from the properties of unions). In fact we have the following.

Corollary 25. *A topological space X is sober if and only if all completely prime filters on $\Omega(X)$ are of the form $\{U \in \Omega(X) : x \in U\}$ for some $x \in X$.*

Proof. First suppose X is sober, and let $F \subseteq \Omega(X)$ be a completely prime filter. Then, by the proof of the above theorem, we have that $\mu_{\Omega X}(F)$ is meet-irreducible, so that by assumption there is an $x \in X$ such that $\mu_{\Omega X}(F) = X \setminus \overline{\{x\}}$. Applying $\tilde{\mu}_{\Omega X}$ on both sides we obtain

$$F = \{U \in \Omega(X) : U \nsubseteq X \setminus \overline{\{x\}}\}.$$

Observe that $X \setminus \overline{\{x\}} = \operatorname{int}\{x\}^c = \bigcup\{V \in \Omega(X) : x \notin V\}$, so that $U \nsubseteq \bigcup\{V \in \Omega(X) : x \notin V\}$ iff $x \in U$, whence

$$F = \{U \in \Omega(X) : x \in U\}.$$

Conversely, suppose each completely prime filter on $\Omega(X)$ is of the form given in the statement of the corollary, and let $M \in \Omega(X)$ be meet-irreducible. As before, $\tilde{\mu}_{\Omega X}(M)$ is a completely prime filter, so that there is an $x \in X$ with $\tilde{\mu}_{\Omega X}(M) = \{U \in \Omega(X) : x \in U\}$. Applying $\mu_{\Omega X}$ on both sides yields

$$M = \bigcup\{U \in \Omega(X) : x \notin U\} = \operatorname{int}\{x\}^c = X \setminus \overline{\{x\}}.$$

□

We thus have three perspectives on points. The analogy between our original definition of a point and points of a topological space is clear. If we think of points as in the beginning of this section, namely, as areas of space diminishing in size, we are thinking of a point as a filter. In this context, a useful way to think about elements of a filter is as pieces of evidence (e.g. outcomes of a measurement) which tell us with variable precision where the 'real value' lies. This is one of the motivations behind studying point-free topology, as there may be situations in which the 'real value' is inaccessible or does not exist to begin with. Then the structure of a filter has an epistemic interpretation: if x is a piece of evidence for the ideal point, then so is any piece of evidence containing x; if x and y contain the ideal point, then so does $x \wedge y$; zero (the inconclusive or empty evidence) is evidence for no point. Complete primeness can be given the interpretation of consistency of parts: if some pieces of evidence put together contain the ideal point, then one of the pieces must contain it (so we exclude the situation in which the whole contains the ideal point but none of the parts do). Natural bijection with meet-irreducible elements is reminiscent of the fact that a point is determined by its complement.

A slight modification of the functor C turns it into a functor from **Loc** to **Top**. We just have to topologise the set $C(L)$ in such a way that $(f^*)^{-1}$ is continuous for any $f : L \to K$. Given a locale L and an element $a \in L$, we define Σ_a as the collection of all completely prime filters containing a:

$$\Sigma_a \coloneqq \{F \in C(L) : a \in F\}.$$

Observe that $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b}$ and moreover $\Sigma_0 = \varnothing$ and $\Sigma_1 = C(L)$ (we require any filter to be non-empty). Thus the set $\tau(L) \coloneqq \{\Sigma_a : a \in L\}$ forms a basis for a topology on $C(L)$. In fact, $\tau(L)$ is closed under unions: using complete primeness of the filters it is straightforward to see that

$$\bigcup_{a \in A} \Sigma_a = \Sigma_{\bigvee A}$$

for any subset $A \subseteq L$. Thus $\tau(L)$ defines a topology on $C(L)$. Then, we have the following.

Proposition 26. *Let $f : L \to K$ be a localic map. Then*

$$(Cf)^{-1}\Sigma_y = \Sigma_{f^*(y)}$$

for any $y \in K$.

Proof. For a completely prime filter $F \subseteq L$ we have: $F \in (Cf)^{-1}\Sigma_y$ iff $(f^*)^{-1}F \in \Sigma_y$ iff $y \in (f^*)^{-1}F$ iff $f^*(y) \in F$ iff $F \in \Sigma_{f^*(y)}$. □

It follows that $Cf : C(L) \to C(K)$ is a continuous map when $C(L)$ and $C(K)$ are equipped with topologies $\tau(L)$ and $\tau(K)$. We thus define

$$\begin{aligned} \Sigma : \mathbf{Loc} &\to \mathbf{Top} \\ L &\mapsto (C(L), \tau(L)) \\ \left(L \xrightarrow{f} K\right) &\mapsto \left(\Sigma(L) \xrightarrow{(f^*)^{-1}} \Sigma(K)\right), \end{aligned}$$

where $(C(L), \tau(L))$ denotes the topological space $C(L)$ equipped with the topology $\tau(L)$. Note that on the level of sets, Σ is identical to C, and the only way it differs from it is that we equip the $C(L)$ with a topology. To put precisely, we have $U\Sigma = C$, where $U : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor.

Since completely prime filters of a locale are in a natural one-to-one correspondence with points of the locale, we may think of $\Sigma(L)$ as the set of points of L together with the topology induced on it by L.

For each $L \in \mathbf{Loc}$, we define a map

$$\sigma_L : \Omega\Sigma(L) \to L$$

as the right adjoint to the map

$$\begin{aligned} \sigma_L^* : L &\to \Omega\Sigma(L) \\ a &\mapsto \Sigma_a. \end{aligned}$$

Since $\Sigma_{\bigvee A} = \bigcup_{a \in A} \Sigma_a$ for all subsets $A \subseteq L$, we have that σ_L^* preserves joins and thus by the Adjoint functor theorem for posets it has a right adjoint given by

$$\sigma_L(U) = \bigvee \{a \in L : \Sigma_a \subseteq U\}.$$

Since $\Sigma_1 = C(L)$ and $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$, we have that σ_L^* preserves finite meets, so that σ_L is indeed a localic map.

Remark 27. Observe that σ_L^* is surjective. Using the adjoint equation we have that $x \leq \sigma_L \sigma_L^*(x)$ and $\sigma_L^* \sigma_L(U) \leq U$ for all $x \in L$ and $U \in \Omega\Sigma(L)$, from which it is immediate that

$$\sigma_L^* \sigma_L \sigma_L^* = \sigma_L^*,$$

and thus by surjectivity of σ_L^* we obtain $\sigma_L^* \sigma_L = \mathrm{id}_{\Omega\Sigma L}$. Thus *every σ_L is injective.*

Proposition 28. *The assignment*

$$\sigma : \Omega\Sigma \to \mathrm{id}_{\mathbf{Loc}},$$

whose components are given by σ_L, is a natural transformation.

Proof. We need to show that for any localic map $f : L \to K$, the diagram

$$\begin{tikzcd} \Omega\Sigma(L) \arrow[r, "(Cf)_*"] \arrow[d, "\sigma_L"'] & \Omega\Sigma(K) \arrow[d, "\sigma_K"] \\ L \arrow[r, "f"'] & K \end{tikzcd}$$

commutes.This amounts to showing that for all $b \in L$ we have

$$f\left(\bigvee\{a \in L : \Sigma_a \subseteq \Sigma_b\}\right) = \bigvee\{y \in K : \Sigma_y \subseteq \bigcup\{\Sigma_w : \Sigma_{f^*(w)} \subseteq \Sigma_b\}\}.$$

We first observe that for all $y \in K$,

$$\Sigma_y \subseteq \bigcup\{\Sigma_w : \Sigma_{f^*(w)} \subseteq \Sigma_b\} \text{ iff } \Sigma_{f^*(y)} \subseteq \Sigma_b.$$

Indeed, the right-to-left implication is immediate. Conversely, suppose

$$\Sigma_y \subseteq \bigcup\{\Sigma_w : \Sigma_{f^*(w)} \subseteq \Sigma_b\}$$

and let $F \in \Sigma_{f^*(y)}$ so that $f^*(y) \in F$, or equivalently $y \in (f^*)^{-1}F$. By Proposition 22, this is a completely prime filter, so that $(f^*)^{-1}F \in \Sigma_y$. By assumption, there is a $w \in K$ such that $\Sigma_{f^*(w)} \subseteq \Sigma_b$ and $(f^*)^{-1}F \in \Sigma_w$. The latter occurs precisely when $F \in \Sigma_{f^*(w)}$, whence it follows that $F \in \Sigma$, showing the left-to-right entailment.

Thus what we have to show simplifies to

$$f\left(\bigvee\{a \in L : \Sigma_a \subseteq \Sigma_b\}\right) = \bigvee\{y \in K : \Sigma_{f^*(y)} \subseteq \Sigma_b\}. \tag{29}$$

Hence let $y \in K$ be such that $\Sigma_{f^*(y)} \subseteq \Sigma_b$, so that

$$f^*(y) \leq \bigvee\{a \in L : \Sigma_a \subseteq \Sigma_b\} \text{ iff } y \leq f\left(\bigvee\{a \in L : \Sigma_a \subseteq \Sigma_b\}\right)$$

using the adjoint equation. Thus the left-hand side of (29) is an upper bound for the set $\{y \in K : \Sigma_{f^*(y)} \subseteq \Sigma_b\}$. To see that it is the least upper bound, let $u \in K$ be an upper bound, so whenever $\Sigma_{f^*(y)} \subseteq \Sigma_b$, we have $y \leq u$. Using the fact that for any $a \in L$ we have $f^*f(a) \leq a$, and that σ_L^* is order-preserving, we get that

$$\begin{aligned}
\Sigma_{f^*f(\bigvee\{a\in L:\Sigma_a\subseteq\Sigma_b\})} &\subseteq \Sigma_{\bigvee\{a\in L:\Sigma_a\subseteq\Sigma_b\}} \\
&= \bigcup\{\Sigma_a : a \in L \text{ and } \Sigma_a \subseteq \Sigma_b\} \\
&= \Sigma_b.
\end{aligned}$$

Then, by assumption, $f\left(\bigvee\{a \in L : \Sigma_a \subseteq \Sigma_b\}\right) \leq u$, so that the left-hand side of (29) is indeed the supremum, proving the desired equality (29). $\square$

Now we define a continuous map for each $X \in \mathbf{Top}$

$$\begin{aligned} \lambda_X : X &\to \Sigma\Omega(X) \\ x &\mapsto \{U \in \Omega(X) : x \in U\}. \end{aligned}$$

To see that it is indeed continuous, let $U \subseteq X$ be open; we then have

$$x \in \lambda_X^{-1}(\Sigma_U) \text{ iff } U \in \{W \in \Omega(X) : x \in W\} \text{ iff } x \in U,$$

so that $\lambda_X^{-1}(\Sigma_U) = U$.

Proposition 30. *The assignment*

$$\lambda : \mathrm{id}_{\mathbf{Top}} \to \Sigma\Omega,$$

whose components are given by λ_L, is a natural transformation.

Proof. We need to show that for any continuous map $f : X \to Y$, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X"'] & Y \arrow[d, "\lambda_Y"] \\ \Sigma\Omega(X) \arrow[r, "(f^{-1})^{-1}"'] & \Sigma\Omega(Y) \end{tikzcd}$$

commutes. This amount to showing that for each $x \in X$

$$\lambda_Y(f(x)) = (f^{-1})^{-1}\lambda_X(x),$$

which is straightforward; for given an open subset $W \subseteq Y$ we have

$$\begin{aligned} W \in (f^{-1})^{-1}\lambda_X(x) \quad &\text{iff} \quad f^{-1}W \in \lambda_X(x) \\ &\text{iff} \quad x \in f^{-1}W \\ &\text{iff} \quad f(x) \in W \\ &\text{iff} \quad W \in \lambda_Y(f(x)). \end{aligned}$$

$\square$

We are now ready to state and prove the main result of this section.

Theorem 31. *The functors*

$$\Omega : \mathbf{Top} \rightleftarrows \mathbf{Loc} : \Sigma$$

are adjoint (left adjoint on the left), with unit λ and counit σ.

Proof. For any $X \in \mathbf{Top}$ and $L \in \mathbf{Loc}$, we need to show that the triangles

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega\lambda_X"] \arrow[dr, "\mathrm{id}_{\Omega X}"'] & \Omega\Sigma\Omega X \arrow[d, "\sigma_{\Omega X}"] \\
& \Omega X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma L \arrow[r, "\lambda_{\Sigma L}"] \arrow[dr, "\mathrm{id}_{\Sigma L}"'] & \Sigma\Omega\Sigma L \arrow[d, "\Sigma\sigma_L"] \\
& \Sigma L
\end{tikzcd}
\tag{32}$$

commute.

For the left triangle, let $U \subseteq X$ be an open subset. We need to show that

$$\bigcup\{V \in \Omega(X) : \Sigma_V \subseteq (\lambda_X)_* U\} = U.$$

Note that it suffices to show that for any open $V \subseteq X$,

$$\Sigma_V \subseteq (\lambda_X)_* U \quad \text{iff} \quad V \subseteq U,$$

which we now do. We have that

$$\begin{aligned}
\Sigma_V \subseteq (\lambda_X)_* U \quad &\text{iff} \quad V \in F \implies F \in \bigcup\{\Sigma_W : W \in \Omega(X) \text{ and } \lambda_X^{-1}\Sigma_W \subseteq U\} \\
&\text{iff} \quad V \in F \implies \exists W \in \Omega(X) \text{ s.t. } F \in \Sigma_W \text{ and } \lambda_X^{-1}\Sigma_W \subseteq U \\
&\text{iff} \quad V \in F \implies \exists W \in \Omega(X) \text{ s.t. } W \in F \text{ and } W \subseteq U \\
&\text{iff} \quad V \subseteq U,
\end{aligned}$$

where whenever we write $V \in F$, there is an implicit universal quantification over $F \in \Sigma\Omega(X)$, which we omitted for the sake of clarity. For the third equivalence we used the observation that $\lambda_X^{-1}\Sigma_W = W$ made when proving continuity of λ_X; and for the last one the fact that $\{W \in \Omega(X) : x \in W\}$ is a completely prime filter.

For the right triangle, we need to show that for any completely prime filter F on L we have

$$(\sigma_L^*)^{-1}\{\Sigma_a : a \in L \text{ and } F \in \Sigma_a\} = F.$$

But the set of which we take the preimage on the left-hand side is equal to $\{\Sigma_a : a \in F\}$; thus we have for all $b \in L$

$$b \in (\sigma_L^*)^{-1}\{\Sigma_a : a \in F\} \quad \text{iff} \quad \Sigma_b \in \{\Sigma_a : a \in F\} \quad \text{iff} \quad b \in F,$$

where the 'only if' part of the last equivalence follows by noting that if $\Sigma_b = \Sigma_a$, then $b \in F$ iff $a \in F$. □

In the light of Proposition 10, it is natural to ask what are the categories the above adjunction restricts to. We conclude this section by addressing this question.

Proposition 33. *The map $\sigma_L : \Omega\Sigma(L) \to L$ is an isomorphism if and only if L is a spatial locale.*

Proof. If σ_L is an isomorphism, L is isomorphic to $\Omega\Sigma(L)$ and hence spatial.

For the converse, first observe that for any topological space, the map $\sigma_{\Omega X}$ is an isomorphism: it is injective by Remark 27, surjective by the left triangle in (32), and hence an isomorphism by Proposition 17. Thus if there is a topological space X and an isomorphism $L \xrightarrow{\sim} \Omega(X)$, then naturality of σ together with the fact that $\sigma_{\Omega X}$ is an isomorphism imply that so is σ_L. □

In light of the above proposition, the following characterisation (making no reference to topological spaces) of spatial locales becomes easy to see (Johnstone [2, p. 43]).

Corollary 34. *A locale L is spatial if and only if any elements $a, b \in L$ with $a \not\leq b$ can be separated by a point (meaning there is a completely prime filter containing one but not the other).*

Proof. If L is spatial, then this is immediate, as for any (open) sets U and V with $U \nsubseteq V$, there is an $x \in U$ such that $x \notin V$, so that $\{W : x \in W\}$ is the separating filter.

To show the converse, it suffices to see that σ_L^* is injective (by Remark 27 and Proposition 17). Hence suppose $\Sigma_a = \Sigma_b$, so that $a \in F$ iff $b \in F$ for all completely prime filters F. By assumption, this can only happen if $a \leq b$ and $b \leq a$, so that $a = b$, as required. □

Proposition 35. *The map $\lambda_X : X \to \Sigma\Omega(X)$ is a homeomorphism if and only if X is sober.*

Proof. We use the formulation of sobriety in Corollary 25.

If λ_X is an isomorphism and $F \subseteq \Omega(X)$ is a completely prime filter, then

$$\{U \in \Omega(X) : \lambda_X^{-1}(F) \in U\} = \lambda_X \lambda_X^{-1}(F) = F.$$

Conversely, if X is sober, then each element in $\Sigma\Omega(X)$ is of the form $\{U \in \Omega(X) : x \in U\}$ for some $x \in X$. Moreover, such x is in fact unique, since we are assuming the spaces to be T_0. Thus λ_X is bijective. It thus remains to show it is also an open map. Thus let $U \subseteq X$ be open. We have

$$\lambda_X(U) = \{\{V \in \Omega(X) : x \in V\} : x \in U\} = \{F \in \Sigma\Omega(X) : U \in F\} = \Sigma_U.$$

□

Let us denote the full subcategory of **Loc** consisting of spatial locales by **SpLoc** and the full subcategory of **Top** containing the sober spaces by **Sob**. As anticipated, we have the following result.

Theorem 36. *The adjunction in Theorem 31 restricts to the equivalence of categories*

$$\mathbf{Sob} \simeq \mathbf{SpLoc}.$$

Proof. Apply Proposition 10 to propositions 33 and 35. □

5 Sublocales

Here we define the appropriate notion of a *sublocale* and scratch the surface of the structure of sublocales of a given locale. In particular, we shall see that the collection of all sublocales of a locale form a colocale, and hence a co-Heyting algebra.

Definition 37 (Sublocale)**.** Let L be a locale. A subset $S \subseteq L$ is a *sublocale* of L if it is a locale in the induced order and the embedding map $j : S \hookrightarrow L$ is localic.

Proposition 38. *A subset $S \subseteq L$ of a locale is a sublocale if and only if the following conditions hold:*

(S1) S is closed under meets in L,

(S2) for every $s \in S$ and $x \in L$, we have $x \to s \in S$.

Proof. First suppose S is a sublocale and let $j : S \hookrightarrow L$ be the embedding. For (S1), let $A \subseteq S$. Then

$$\bigwedge A = j\left(\bigwedge_S A\right) = \bigwedge_S A \in S$$

since j preserves meets. This implies that the Heyting implication in S coincides with the Heyting implication in L. For (S2), let $s \in S$ and $x \in L$. Using (S1) we compute for any $y \in L$

$$\begin{aligned} y \leq x \to s \quad &\text{iff} \quad y \wedge x \leq s \\ &\text{iff} \quad y \wedge x \leq j(s) \\ &\text{iff} \quad j^*(y) \wedge j^*(x) \leq s \\ &\text{iff} \quad j^*(y) \leq j^*(x) \to s \\ &\text{iff} \quad y \leq j^*(x) \to s, \end{aligned}$$

whence we conclude $x \to s = j^*(x) \to s \in S$.

Conversely, suppose (S1) and (S2) hold. Thus given $A \subseteq S$, we have $\bigwedge A \in S$, so that it is also the infimum of A in S. Thus S is a complete lattice. Since S is closed under the Heyting implication, it is a complete Heyting algebra and hence a locale by Proposition 13. It is now immediate that the embedding j preserves meets, as those in S coincide with those in L. It remains to show that its left adjoint $j^* : L \to S$ preserves finite meets. First, for $s \in S$,

$$j^*(1) \leq s \quad \text{iff} \quad 1 \leq j(s) = s,$$

whence $j^*(1) = 1$. For $a, b \in L$ and $s \in S$ we compute

$$\begin{aligned} j^*(a \wedge b) \leq s \quad &\text{iff} \quad a \wedge b \leq s \\ &\text{iff} \quad a \leq b \to s \\ &\text{iff} \quad j^*(a) \leq b \to s \\ &\text{iff} \quad b \leq j^*(a) \to s \\ &\text{iff} \quad j^*(b) \leq j^*(a) \to s \\ &\text{iff} \quad j^*(b) \wedge j^*(a) \leq s, \end{aligned}$$

where in the third and fifth equivalence we used (S2). Thus we conclude that $j^*(a \wedge b) = j^*(a) \wedge j^*(b)$, as required. $\square$

Using the characterisation of sublocales in Proposition 38, we immediately see that the top element of L is contained in any sublocale, $\{1\}$ and L are always a sublocales and any intersection of sublocales is itself a sublocale. Thus for any locale L the poset of its sublocales $\mathcal{S}\ell(L)$ (ordered by inclusion) is a complete lattice with bottom element $\{1\}$, top element L, and whose meets are given by intersections. The joins are given by

$$\bigvee \mathfrak{S} = \left\{ \bigwedge A : A \subseteq \bigcup \mathfrak{S} \right\},$$

where $\mathfrak{S}$ is a collection of sublocales (see Picado and Pultr [5, p. 28] for details). In fact we have the following theorem, whose proof may be found in Picado and Pultr [5, Theorem 3.2.1].

Theorem 39. *For any lattice L, the poset of its sublocales $\mathcal{S}\ell(L)$ is a colocale, that is, it is a complete lattice whose finite joins distribute over arbitrary meets.*

In any lattice D, the fact that finite joins distribute over arbitrary meets may be expressed as the coproduct functors $a \vee -$ preserving meets, thus by the Adjoint functor theorem for posets 7 it has a left adjoint $- \rightsquigarrow a$ satisfying

$$c \rightsquigarrow a \leq b \quad \text{iff} \quad c \leq a \vee b$$

for all $a, b, c \in D$. This is the *co-Heyting implication* exhibiting D as a co-Heyting algebra[2]. For future reference, we define the *co-Heyting negation* of $a \in D$ by $\neg a \coloneqq 1 \rightsquigarrow a$, where 1 is the top element of D. Observe that we have $a \vee \neg a = 1$ for all $a \in D$, but in general we need not have $a \wedge \neg a = 0$.

By analogy with the topological spaces, we wish to talk about open and closed sublocales. For the motivation behind these definition see Picado and Pultr [5, p. 33].

Definition 40. Let L be a locale and $a \in L$. Then the *open sublocale* associated with a is

$$\mathfrak{o}(a) \coloneqq \{a \to x : x \in L\};$$

and the *closed sublocale* associated with a is

$$\mathfrak{c}(a) \coloneqq \uparrow a.$$

[2]Note that the situation is dual to that of Proposition 13.

It is straightforward to see that both open and closed sublocales associated with a are indeed sublocales. By considering the sublocales associated with the top element, we get that both L and $\{1\}$ are both open and closed. Furthermore, by analogy with closed and open subspaces, we have the following.

Proposition 41. *Let L be a locale. Then for any $a \in L$, the sublocales $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other in $\mathcal{S}\ell(L)$.*

Proof. First let $x \in L$. Using that in any Heyting algebra we have $x = (x \vee a) \wedge (a \to x)$, we observe that $x \in \mathfrak{c}(a) \vee \mathfrak{o}(a)$ since $(x \vee a) \in \mathfrak{c}(a)$ and $(a \to x) \in \mathfrak{o}(a)$. Thus $L \subseteq \mathfrak{c}(a) \vee \mathfrak{o}(a)$.

Next observe that $y \in \mathfrak{c}(a) \cup \mathfrak{o}(a)$ implies $a \leq y$ and $y = a \to z$ for some $z \in L$, so that $a \leq z$, whence $y = 1$. Thus $\mathfrak{c}(a) \cup \mathfrak{o}(a) = \{1\}$. $\square$

The open and closed sublocales behave in the expected way under meets and joins.

Proposition 42. *[5, Proposition 6.1.5] We have*

$$\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b), \qquad \bigvee_{i \in I} \mathfrak{o}(a_i) = \mathfrak{o}\left(\bigvee_{i \in I} a_i\right),$$

$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \qquad \bigcap_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}\left(\bigvee_{i \in I} a_i\right).$$

We conclude this section by defining closure and interior of sublocales. This may be done in exactly the same way as for spaces: if $S \subseteq L$ is a sublocale, the interior of S is the largest open sublocale contained in S, and the closure of S is the smallest closed sublocale containing S (which exist since $\mathcal{S}\ell(L)$ is a complete lattice). We observe that for closure we in fact have a simpler description: any closed sublocale containing S must also contain $\bigwedge S$ as an element, and consequently contains $\uparrow(\bigwedge S)$. On the other hand $\uparrow(\bigwedge S)$ is closed and contains S, whence we conclude

$$\overline{S} = \uparrow\left(\bigwedge S\right).$$

Interior does not admit such a nice description, however, we do have

$$\begin{aligned} \mathrm{int}S &= \bigvee\{\mathfrak{o}(x) \subseteq S : x \in L\} \\ &= \mathfrak{o}\left(\bigvee\{x \in L : \mathfrak{o}(x) \subseteq S\}\right) \\ &= \left\{\bigwedge_{x \in A} (x \to y) : y \in L\right\}, \end{aligned}$$

where $A \coloneqq \{x \in L : \forall z \in L(x \to z \in S)\}$.

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